The Effects of Uncertainty and Irreversibility on Investment

Koichi Mashiyama
Department of Economics
Meiji Gakuin University
Discussion Paper No.09-01

March, 2009: preliminary version

Abstract

The present paper examines the robustness of the result derived in the canonical model of investment under uncertainty, and explores the relationship between uncertainty and investment. We have arrived upon the conclusion that, for three different cases of the demand shock, whether or not an increase in uncertainty depresses investment depends on the concavity of the operating profit function with respect to the demand shock. However even if the operating profit function is convex in the demand shock, an increase in uncertainty may not necessarily raise investment and would depress investment, depending on the range of model parameters. This result implies that the convexity of the operating profit function need not necessarily to be dismissed due to the emprirical validity of the model.

1 Introduction

The effects of uncertainty on investment have long been extensively explored in the literature on the investment theory. Taken as whole, different theories emphasize different channels, some pointing to a positive relationship and others to a negative relationship. We can classify theories of investment under uncertainty into two strands of theory. One focuses on the adjustment cost and the other emphasizes the irreversibility of investment.

In the models of Hartman(1972), Abel(1983,85), Abel and Eberly(1994), the marginal revenue product of capital is convex and it is the flexibility of labor relative to capital that generates this convexity. When the operating profits are convex in the shock against the market demand, any mean-preserving increases in the distribution of the shock raises investment of the firm. On the other hand, Caballero(1991) and Pindyck(1993) pointed out that the relationship between uncertainty and investment should not be expected from the adjustment costs literature alone and when the firm faces an elastic demand curve in an imperfectly competitie market, the negative relation between investment and uncertainty is likely to be reversed [1].

In the canonical models of investment with irreversibility based on the real options approach, as shown in Bernanke(1983), McDonald and Siegel(1986), Pindyck (1988), Bertola and Caballeo(1994), and Dixit and Pindyck(1994), an increase in the volatility of price flucuations leads to a rise of trigger point to invest and then decelerates investment. In this literature,

[1] See also Caballero and Pindyck(1996), Abel and Eberly(1996,1999).

uncertainty affects irreversible investment in two ways: first, through the effects of the risk premium component on the marginal profitability of capital, and second, through the effects on the tigger threshold of the value of waiting [2].

Those models mentioned above predict a positive or negative effect of uncertainty on investment depending on whether the marginal revenue product of capital is a convex or concave function of the exogenous shock. In the case of convexity, an increase in the variance of the shock would raise investment via Jensen's inequality. Leahy and Whited(1996) showed in their empirical studies that uncertainty exerts a strong negative influence on investment and that uncertainty affects investment directly rather than working through covariances. This result casts doubt on the importnace of theories that emphasize the convexity of the marginal revenue product of capital such as a Hartman-Abel model.

The present paper examines the robustness of the conclusion derived in the canonical model of investment under uncertainty and explore the relationship between uncertainty and investment. A canonical model of investment under uncertainty is formulated in Section 2. The section 3 is devoted to analyzing the relationship between uncertainty and investment in this canonical model. We explore the investment behavior with three different dynamical systems governing the shock processes: first, when the demand shock is governed by the familiar geometric Brownian motion; second, when it is governed by a mean-reverting stochastic process, and third, when it obeys a geometric Brown-Poisson process.

2 A Canonical Model of Investment under uncertainty

We consider a firm that produces output using capital K and variable factors of production. It earns the operating profit that depends on the random variable X. The operating profit at time t is denoted by $\pi(K(t), X(t))$. $K(t)$ is the capital stock at time t, and $X(t)$ is an exogenous shock to the productivity, factor prices, or the demand for products.

We consider the standard case that the production function is given by the following Cobb-Douglas production function:

$$y = F(L, K) = L^{\alpha} K^{1-\alpha},\ 0 < \alpha < 1. \tag{1}$$

The demand curve faced by the firm is assumed to be given by

$$P(t) = Q(t)^{(1-\phi)/\phi} X(t)^{\varphi}, \phi \geq 1,\ \varphi > 0, \tag{2}$$

where, $P(t)$ is the market price at time t, $Q(t)$ is the aggregate quantity of product supplied in the market. When the product market is perfectly competitive, we set $\phi = 1$. When $\varphi = 1$, the demand shock is proportional to the market price, so that the demand shock is interpreted as a shock to the market price. Assume that $Q = Ny$ ($N = 1$ is the number of firms), and then the operating profit $\pi(K, X) = P(t)L^{\alpha}K^{1-\alpha} - wL(t)$ is given by

$$\pi(K(t), X(t)) = hX(t)^{\nu} K(t)^{\beta}, \tag{3}$$

[2] Recently this negative relationship between uncertainty and investment is questioned. Sarkar(2000), Gryglewicz et al(2008) and Wong(2007) have suggested that an increase in the trigger threshold may not induce a delay in the timming of investment, i.e. the investment-uncertainty relationship is not neccessarily monotonic.

where, w is the wage rate and

$$
\begin{aligned}
h &= (1-\frac{\alpha}{\phi})(\frac{\alpha}{\phi}\frac{1}{w})^{(\frac{\alpha}{\phi})/(1-\frac{\alpha}{\phi})}, \\
\nu &= \frac{\varphi\phi}{\phi-\alpha} > 0, \\
\beta &= \frac{1-\alpha}{\phi-\alpha} \le 1.
\end{aligned}
$$

Here, it is clear that $\pi_K > 0, \pi_{KK} \ : \le 0, \pi_X > 0$. The operating profit is a concave function of capital stock K. If $\nu > 1$, the profit function is a convex function of the exogenous shock X. When $\nu < 1$, the profit function is a concave function of the exogenous shock. $\nu > 1$ if and only if $\varphi > 1 - \alpha/\phi$. This fact implies that the marginal revenue product of capital is a decreasing function of capital but convex or concave in the value of the exogenous shock depending on the value of the parameter φ [3]. When the market is perfetly competitive, the profit function depends linearly on the capital stock K. On the other hand, when the market is imperfetly competitive, the profit function is strictly concave in the capital stock K.

We utilize three different stochastic differential equations for the exogenous shock processes: the familiar geometric Brownian motion, a mean-reverting stochastic process, and a geometric Brown-Poisson process.

The geometric Brownian case. The random variable X is governed by the following stochatic differential equation:

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dz, \tag{4}$$

where z is the standard Brownian motion. The random variable X obeys the geometric Brownian motion if the drift and vairance coefficients are expressed by

$$\mu(X) = \mu X, \ \sigma(X) = \sigma X.$$

The mean-reverting case. The random variable X is governed by the following mean-reverting differential equation:

$$dX(t) = \iota(\mu - X(t))dt + \sigma X(t)dz(t), \tag{5}$$

where, ι is the speed of reversion, μ is the long-term demand level, and σ is the volatility of the process.

The geometric Brown-Poisson case. The random variable X is governed by the following geometric Brown-Poisson differential equation:

$$dX = (\mu - \tilde{\lambda}k)X(t)dt + \sigma X(t)dz(t) + kX(t)dN(t), \tag{6}$$

where N is the Poisson process, $\tilde{\lambda}$ is the arrival rate or intensity rate of Poisson process, and k is the expected amplitude of size on jump.

The capital stock grows according to

$$dK(t) = (I(t) - \delta K(t))dt, \tag{7}$$

[3] $\nu = 1$ is assumed in deriving their conclusion in Abel(1983), Abel and Eberly(1994), and Caballero and Pindyck(1996). The specification that $\nu < 1$ coupled with the linear homogeneity of π in K and X is presupposed in the model of Abel and Eberly(1996,199). Caballero(1991) used the presupposition that $\nu > 1$.

where, $I(t)$ is the investment at time t, δ is the rate of depreciation. In order to build a new capital stock, the adjustment cost is inquired. The direct cost of investment is composed of the purchase/sale price of capital goods and the installing/detaching costs. The adjustment costs of investment arise from training workers and expanding the operating capacity to manage and operate the plant and machines at the larger scale. The fixed cost is independent of the amount of investment but this cost may be linearly homogeneous of investment and capital if it reflects stopping the operation and/or starting up the plant conditions in the efficient way and on the different scale. The gross cost of investment can be therefore expressed in the function form $c(I, K)$. The gross investment cost function $c(I, K)$ is assumed strictly convex for I and continuous except for at the origin $I = 0$. The right-hand derivative on the origin is assumed to be larger than the left-hand derivative on that point. That is,

$$\lim_{h\to 0} c_I(0+h, K) \geq \lim_{h\to 0} c_I(0-h, K), \quad h > 0$$

This assumption implies that the sale of capital goods cannot be accomplished at the same price as the their purchase and there are also installation costs, which are added to the purchase price but cannot be recovered on sale. There may be additional costs of detaching and moving to other places, and sufficiently specialized machinery and plants may have little value to others.

We assume that the firm is risk neutral or risk averse and maximizes the expected present value of $\pi(K, X)$ minus $c(I, K)$. The present value V of the firm is given by

$$V(K(s), X(s)) = \max_I E_s[\int_0^\infty \{\pi(K(t+s), X(t+s)) - c(I(t+s), K(t+s))\}e^{-\rho t}dt] \tag{8}$$

where, E_s is the conditional expectation operator at time s, and $\rho > 0$ is the discount rate that the investors or stockhholders require. Using dynamic programming technique, we have

$$\rho V(K, X) = \max_I \{\pi(K, X) - c(I, K) + \frac{1}{dt}E_s dV.\} \tag{9}$$

The left hand side is the required rate of return and the right hand side is the maximized rate of return on investment. The maxizimed rate of return is composed of the gross profit $\pi(K, X) - c(I, K)$ and the capital gain EdV/dt. Applying the Ito's lemma, we have the Hamilton-Jacobi equation

$$V_s + \max_I[\mathcal{A}(s)V + \pi(K, X) - c(I, K)] = \rho V(K, X),$$

where $\mathcal{A}$ is the infinitesimal generator (second order differential operator). When the stochastic differential equation (4) is used, the infinitesimal generator is given by [4]

$$\mathcal{A}(s)V = (I - \delta K)V_K + \mu(X)V_X + \frac{1}{2}\sigma(X)^2 V_{XX}.$$

Since $V_s = 0$,

$$\max[\pi(K, X) - c(I, K) + (I - \delta K)V_K + \mu(X)V_X + \frac{1}{2}\sigma(X)^2 V_{XX}] = \rho V(K, X), \tag{10}$$

[4]The infinitesimal generator for the stochastic differential equation (5) or (6) is similarly given. The analysis for these cases will be conducted in the next section.

Here, f_x denotes the partial derivative of f by x, f_{xx} is the second order partial derivative by x. The marginal value q of the firm is defined by $q = V_K$. Optimal investment I^* is the amount of investment which maximizes the function:

$$\psi(I; K, X) \equiv IV_K - c(I, K)$$

Optimal investment depends on $q(= V_K)$ and K. Therefore, we can express this relationship by $I^* = I^*(q, K)$.

If the gross investment cost function is differentiable, the first order condition is given by

$$c_I(I^*, K) = q.$$

It should be reminded that the function $c(I, K)$ is not differentiable at the origin and $c_I(0, K)^+ > c_I(0, K)^-$. Since $c(I, K)$ is not differentiable at $I = 0$, optimal investment must be zero as far as $c_I(0, K)^+ \geq q \geq c_I(0, K)^-$ is satisfied. Hence optimal investment is characterized as follows:

$$\begin{array}{ll} c_I(I^*, K) = q, & \text{when } q > c_I(0, K)^+, \\ I^* = 0, & \text{when } c_I(0, K)^+ \geq q \geq c_I(0, K)^-, \\ c_I(I^*, K) = q, & \text{when } c_I(0, K)^- > q \end{array}$$

When $q > c_I(0, K)^+$, the gross investment is positive but when $q < c_I(0, K)^-$, the gross investment is negative, that is, the old machinery and equipments are on sale. If the investment cost function is kinked at the origin, the investment policy is the trigger strategy of Ss type. The value $q^u = c_I(0, K)^+$ is the upper trigger point and $q^l = c_I(0, K)^-$ is the lower trigger point. Since the investment cost function is assumed strictly convex, investment is an increasing function of q as long as $q > c_I(0, K)^+$. In other word, investment is positively related to the value of q but is not monotonically related.

Next, we derive the formula describing the dynamics of the marginal value of the firm. Taking the derivative of eq.(10) with respect to K, we obtain

$$\pi_K(K, X) - c_K(I^*, K) - \delta q + q_K(I^* - \delta K) + \mu(X)q_X + \frac{1}{2}\sigma(X)^2 q_{XX} = \rho q. \tag{11}$$

It should be noted that q is a function of K and X and so the functional form can be expressed as $q = q(t, K, X)$. Eq.(11) can be expressed in the following form:

$$\mathcal{A}(t)q + \pi_K(K, X) - c_K(I^*, K) - (\rho + \delta)q = 0. \tag{12}$$

Using the Feynman-Kac theorem[5], we have

$$q(t, K, X) = E_t[\int_0^\infty \{\pi_K(K(t+s), X(t+s)) - c_K(I(t+s), K(t+s))\}e^{-(\rho+\delta)s} ds] \tag{13}$$

where we utilized the reasonable condition that $\lim_{T\to\infty} q(T, K, X)e^{-(\rho+\delta)T} = 0$. Thus, q is the present value of the stream of expected marginal profit of capital which consists of the marginal revenue product π_I of investment and the marginal investment cost c_I.

[5]See the textbook on stochstic differential equations, for instance, Karatzas and Shreve(1991), p.364-365.

We assume that the gross adjustment cost of investment can be formulated in the following form:

$$c(I,K)=\begin{cases} a_1K+b_1I+\eta_1 I^{\gamma_1}K^{\gamma_2}, & \text{when } I>0 \\ 0, & \text{when } I=0, \\ a_2K+b_2I+\eta_2|I|^{\gamma_1}K^{\gamma_2}, & \text{when } I<0 \end{cases} \quad \text{(the gross cost function of investment)}$$

where,$a_1, a_2 > 0$, $b_1 > b_2 > 0, \eta_1, \eta_2 \geq 0$. The first terms a_1K, a_2K are interpreted as the fixed cost inquired on investment which depends on caital stock but is independent of the amount of investment. This fixed cost may reflect the cost of stopping and restated while new capital is installed or existing capital is removed [6]. The assumption that $b_1 > b_2 > 0$ captures a partial irreversibility of the past investment. $\eta_2 = \infty$ corresponds to the complete irreversibility. The third term captures the adjustment cost which is commonly assumed linearly homogeneous in the amount of investment and the existing amount of capital stock in the traditional literature. The functional form described above is an extended version used by Abel and Eberly(1994). When $\gamma_1 + \gamma_2 = 1$, the gross investment cost function is linearly homogeneous of investment I and capital K, which implies that optimal investment-capital ratio depends only on the marginal value of capital $q = V_K$. In this case, it can be shown that the marginal $q = V_K$ equals the average q, V/K if the operating profit is a linear function of capital. When $\gamma_2 = 0$, c_K =cosntant, in which case the analytical form of q can be easily derived. We cannot analytically derive the characteristics of optimal investment rule in general and so we need to analyze the probem for two separate cases: the cae I in which $a_1 = a_2 = \gamma_2 = 0$, and the case II in which $\gamma_2 \neq 0$.

3 The relationship between uncertainty and investment

We consider the case I first. The cost function of investment is given by

$$c(I,K)=c(I)=\begin{cases} b_1I+\eta_1 I^{\gamma_1}, & \text{when } I>0 \\ 0, & \text{when } I=0, \\ b_2I+\eta_2|I|^{\gamma_1}, & \text{when } I<0 \end{cases} .$$

Then, $c_K(I,X) = 0$. Optimal investment is characterized by

$$\begin{array}{ll} (I^*)^{\gamma_1-1} = (q-b_1)/(\eta_1\gamma_1) > 0, & \text{when } q > b_1, \\ I^* \;=\; 0, & \text{when } b_1 \geq q \geq b_2, \\ (-I^*)^{\gamma_1-1} = -(q-b_2)/(\eta_2\gamma_1) > 0, & \text{when } b_2 > q \end{array}$$

Eq.(10) is simplified into

$$\max_I \psi(I:K,X) + \pi(K,X) - \delta K V_K + \mu X V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} = \rho V(K,X). \tag{14}$$

Note that when $I^* = 0$, $\max_I \psi(I:K,X) = -c(0)$, and when $I^* \neq 0$,

$$\max_I \psi(I:K,X) = V_K I^* - c(I^*) = V_K(-1)^{i-1}(\frac{V_K - b_i}{\eta_i\gamma_1})^{1/(\gamma_1-1)} - c(I^*),\ i=1,2,$$

[6] This presumption implies that the first terms should be expressed by $a_1\pi(X,K), a_2\pi(X,K)$ where $0 < a_1, a_2 < 1$.

where $i = 1$ if $I > 0$ and $i = 2$ if $I < 0$. Since the marginal value of capital V_K is in general a function of capital stock K, $\max_I \psi(I) = I^* V_K - c(I^*)$ depends on the level of capital stock. It is difficult to derive the analytical solution of the partial differential equation (14) unless the marginal value of capital V_K is independent of capital stock. In other words, the operating profit must be a linear function of capital, which requires the perfect competition in the product market. To obtain the analytical solution, we need to assume that $\beta = 1$.

3.1 The Case I with $\beta = 1$

We assume that $\beta = 1$, i.e. the product market is perfectly competitive. Suppose that $V(K, X) = A(X)K + B(X)$. Substituting this function into the above equation (14), we have

$$[hX^\nu - \delta A(X) + \mu X A'(X) + \frac{1}{2}\sigma^2 X^2 A''(X) - \rho A(X)]K$$
$$+[\max \psi(I) + \mu X B'(X) + \frac{1}{2}\sigma^2 X^2 B''(X) - \rho B(X)] = 0. \tag{15}$$

This relationship must hold for arbitrary values of K and so the value within each bracket must be zero. Hence the following equations hold:

$$hX^\nu - \delta A(X) + \mu X A'(X) + \frac{1}{2}\sigma^2 X^2 A''(X) - \rho A(X) = 0, \tag{16}$$

$$\max \psi(I) + \mu X B'(X) + \frac{1}{2}\sigma^2 X^2 B''(X) - \rho B(X) = 0. \tag{17}$$

The Feynman-Kac formula leads to the solutions

$$A(X(t)) = E_t \int_0^\infty hX(t+s)^\nu e^{-(\rho+\delta)s} ds, \; B(X(t)) = E_t \int_0^\infty \max \psi(I(t+s)) e^{-\rho s} ds. \tag{18}$$

The marginal value of capital is given by

$$q = V_K = A(X).$$

It is obvious that if the market is perfectly competitive, the marginal value of capital does not depend on the existing level of capital stock. Our pressumption is supported. When the market is imperfectly competitive, it is very difficult to obtain an analytical expression for the expected present value $V(K, X)$ of the firm .

To examine the effects of uncertainty on investment, we must specify the nature of the dynamic path governing the state variable X. First we assume that the state variable X obeys the geometric Brownian motion. We can set the expected growth rate of X to be zero without the loss of generality. Then we have

$$E_t[\ln X(t+s)] = \ln X(t) - \frac{1}{2}\sigma^2 s, \; \mathrm{Var}_t[\ln X(t+s)] = \frac{1}{2}\sigma^2 s,$$

so that we have

$$E_t[\ln X(t+s)^\nu] = \nu\{\ln X(t) - \frac{1}{2}\sigma^2 s\},$$
$$\mathrm{Var}_t[\ln X(t+s)^\nu] = \nu^2 \mathrm{Var}_t[\ln X(t+s)].$$

Noting that

$$E_t[X(t+s)^\nu] = \exp[E_t \ln X(t+s)^\nu + \frac{1}{2}\mathrm{Var}_t\{\ln X(t+s)^\nu\}]$$

we have

$$E_t[X(t+s)^\nu] = X(t)^\nu \exp\{\frac{1}{2}\nu(\nu-1)\sigma^2 s\}. \tag{19}$$

Therefore the marginal value of capital q is given by

$$q(t) = V_X = \frac{hX(t)^\nu}{\rho + \delta - \frac{1}{2}\nu(\nu-1)\sigma^2}. \tag{20}$$

This formula is exactly the same result derived by Abel and Eberly(1994), which induces the positive relationship between uncertainty and investment identified in Hartman(1972), Abel(1982), and Caballero(1991) where $\nu > 1$ is presupposed. It is alwas true that $\nu = 1$ if $\varphi = 1$, that is, the shock is a price shock. When $\nu > 1$ (i.e. $\alpha > \phi(\varphi - 1)$), an increase in σ increases the marginal value of capital. In other words, investment is not a decreasing function of σ for given $X(t)$. This positive relationship originates in the convexity of marginal revenue products of capital with respect to $X(t)$ (interpreted as the market prices in the present setting). On the other hand, if $\nu < 1$, an increase in σ decreses the marginal value of capital. In other words, investment is not an increasing function of σ for given $X(t)$. This negative relationship originates in the concavity of marginal revenue products of capital with respect to $X(t)$ (under the condition $\nu < 1$).

In conclusion, an incease in uncertainty raises as well as decreases investment depdending on the values of φ. $\bar{\varphi} = (\phi - \alpha)/\phi$ is the critical value. When $\varphi < \bar{\varphi}$, an increase in uncertainty decelerates investment and othewise, it accelerates investment. There exists no monotonic relationship between uncertainty and investment in the canonical investment model with the convex adjustment cost.

For the model to be consistent with canonical real options models of investment such as developed by Brennan and Schwartz(1985), McDonald and Siegel(1986), and Dixit and Pindyck(1994), the rate of discount ρ used in the preceeding analysis above should be questioned. In any real options models of investment, the discount rate is utilized so as to be consistent with the no-arbitrage condition in asset markets. For instance, CAPM says that

$$\rho = r + \lambda \rho_{im} \sigma,$$

where, λ is the market rice of risk and ρ_{im} is the correlation coefficient between the rate of return on the investment project and the rate of return on the market portfolio in the financial asset market. When we use this relatiohship,

$$q(t) = V_X = \frac{hX(t)^\nu}{r + \lambda\rho_{im}\sigma + \delta - \frac{1}{2}\nu(\nu-1)\sigma^2}. \tag{21}$$

It is easy to see that q is a decreasing function of σ if $\nu < 1$ but when $\nu > 1$ and the following inequality holds

$$\sigma < \bar{\sigma} \equiv \frac{\lambda \rho_{im}}{\nu(\nu-1)},$$

q is a decreasing function of σ. An increase in uncertainty of products demand (price) X decelerates investment if the volatility σ is less than the value $\bar{\sigma}$ but an increase in uncertainty accelerates investment if $\nu > 1$ and σ is greater than the value $\bar{\sigma}$. Since an increase in uncertainty raises the discount rate via the risk premium formula and an increase in the dicount rate leads to a decrease in the present value of future expected profits, otherthings being equal, it comes with little surprise that an increase in uncertainty may decrease the present investment. This effect might be named the discounting effect. Thus, uncertainty may accelerates as well as decelerates investment depending on the value of the volatility of price fluctuations faced the firm. This conclusion is supprted by assuming that the market is perfectly competitive, the production function is linearly homogeneous in labor and capital, and demand fluctuations obey the geometric Brownian motion [7].

What happens when the different stochastic processes are assumed for the dynamical system governing the future pathes of the state variable X. Suppose that the state variable X obeys the following mean-reverting process:

$$dX(t) = \iota(\mu - X(t))dt + \sigma X(t)dz(t), \tag{22}$$

where, ι is the speed of reversion, μ is the long-term demand level, and σ is the volatility of the process. z is the standard Brownian motion. The Hamilton-Jacobi equation (14) can be rewritten into

$$\max_I \psi(I : K, X) + \pi(K, X) - \delta K V_K + \iota(\mu - X(t))V_X + \frac{1}{2}\sigma^2 X^2 V_{XX} = \rho V(K, X). \tag{23}$$

As before, supposing that $V(K, X) = A(X)K + B(X)$ and utilizing the solving technique used above, we have the same formula of the marginal value function:

$$A(X(t)) = E_t[\int_0^\infty hX(t+s)^\nu e^{-(\rho+\delta)s}ds].$$

Note that the solution of the mean-reverting stochastic equation for the state variable X is kwown to be given by [8]

$$X(t+s) = \mu + (X(t) - \mu)e^{-\iota\mu s} + \sigma e^{-\iota\mu s}\int_0^s e^{\iota\mu u}dz(u). \tag{24}$$

The expected value and the variance of X are computed as follows:

$$E_t[X(s)] = \mu + (X(t) - \mu)e^{-\iota\mu s}, \quad \mathrm{Var}_t[X(s)] = \frac{\sigma^2}{2\iota\mu}(1 - e^{-2\iota\mu s}).$$

In the present setting of the model the operating profit function is convex in the demand shock (i.e., $\nu > 1$) or concave in X (i.e. $\nu < 1$). First, to caputure the nature of convexity, we can assume that $\nu = 2$ to simplify our analysis. Then the formula for the marginal value of capital is simplified into

$$q(t)) = \int_0^\infty hE_t[X(t+s)^2]e^{-(\rho+\delta)s}ds.$$

[7] Leahy and Whited(1996) detected the empirical evidence that the discount effect is not a major channel through which uncertainty affects investment. In the subsequent analysis, we will not explicitely discuss about this discount effect.

[8] See Musiela and Rutkowski(2007), Lemma 10.1.2 .

Since

$$E_t[X(t+s)^2] = Var_t[X(t+s)] + [E_t X(t+s)]^2,$$

we obtain

$$\frac{\partial E_t[X(t+s)^2]}{\partial \sigma} = \frac{\sigma}{\iota\mu}(1 - e^{-2\iota\mu s}) > 0.$$

Therefore,

$$\frac{\partial q(t)}{\partial \sigma} = \int_0^\infty h\frac{\sigma}{\iota\mu}(1 - e^{-2\iota\mu s})e^{-(\rho+\delta)s}ds > 0.$$

It is clearly seen that an increase in σ raises the marginal value of capital and so accelerates investment as long as the discounting effect through the risk premium component is ignored. This result comes from the convexity of the profit function with respect to the value of the demand shock. The operating profit is convex if and only if $\varphi > 1 - \alpha/\phi$. When the product market is perfectly competitive, this condition reduces to the inequality: $\varphi > 1 - \alpha$. A sufficient condition for the profit function to be convex is that the fluctuations of the market price are proportional to the demand shock. Hence we can claim that the convexity of the profit function with respect to the demand shock plays the crucial role to assure the positive relationship between the uncertainty and investment.

Now we suppose that the state variable X is governed by the following geometric Brown-Poisson process:

$$dX = (\mu - \tilde{\lambda}k)X(t)dt + \sigma X(t)dz(t) + kX(t)dN(t), \tag{25}$$

where N is the Poisson process, $\tilde{\lambda}$ is the arrival rate or intensity rate of Poisson process, and k is the expected amplitude of size on jump. We assume that $k > -1$. The time path of this stochoastic process is given by

$$X(t) = X(0)\exp\{(\mu - \tilde{\lambda}k - \sigma^2/2)t + \sigma z(t)\}(k+1)^{N(t)}.$$

As well known, the infinitesimal generator for this stochastic differential equation is expressed by[9]

$$\mathcal{A}V(K,X) = V_K(I - \delta K) + V_X(\mu - \tilde{\lambda}k)X + \frac{1}{2}V_{XX}\sigma^2 X^2 + \tilde{\lambda}[V(X(t)) - V(X(t_-))],$$

where, $V(X(t)) - V(X(t_-)) = V((k+1)X(t)) - V(X(t))$. Eq.(14) for the Brown-Poisson case is written by

$$\max_I \psi(I:K,X) + \pi(K,X) - \delta K V_K + V_X(\mu - \tilde{\lambda}k)X + \frac{1}{2}V_{XX}\sigma^2 X^2$$
$$+\tilde{\lambda}[V((k+1)X(t)) - V(X(t))] = \rho V(K,X).$$

Supposing that $V(K,X) = A(X)K + B(X)$, and substituting this function into the above equation, we have

$$[hX^\nu - \delta A(X) + (\mu - \tilde{\lambda}k)XA'(X) + \frac{1}{2}\sigma^2 X^2 A''(X) - (\rho + \tilde{\lambda})A(X) + \tilde{\lambda})A((k+1)X)]K$$
$$+[\max\psi(I) + (\mu - \tilde{\lambda}k)XBA'(X) + \frac{1}{2}\sigma^2 X^2 BA''(X) - (\rho + \tilde{\lambda})B(X) + \tilde{\lambda})B((k+1)X)] = 0.$$

[9]See Shreve(2004), Theorem 11.5.1.

This relationship must hold for arbitrary values of K and so the value within each bracket must be zero. Hence the following equations hold:

$$hX^\nu - \delta A(X) + (\mu - \tilde{\lambda}k)XA'(X) + \frac{1}{2}\sigma^2X^2A''(X) - (\rho + \tilde{\lambda})A(X) + \tilde{\lambda})A((k+1)X) = 0, \quad (26)$$

$$\max \psi(I) + (\mu - \tilde{\lambda}k)XBA'(X) + \frac{1}{2}\sigma^2X^2BA''(X) - (\rho + \tilde{\lambda})B(X) + \tilde{\lambda})B((k+1)X) = 0. \quad (27)$$

Substituting the form $A(X) = aX^\nu$ into the equation (26) gives rise to

$$A(X) = \frac{X^\nu}{\delta + \rho - \mu\nu - \frac{1}{2}\sigma^2\nu(\nu-1) + \tilde{\lambda}\{1 + k\nu - (1+k)^\nu\}}.$$

Since the Poisson process has the variance $\tilde{\lambda}t$ during time period t, the uncertainty arising from the demand shock driven by the Poisson process can be represented by the quantity $\tilde{\lambda}$. It is obvious that if $\nu > 1$, the inequality :

$$1 + k\nu < (1+k)^\nu$$

is satisfied for any positive number k. If $\nu < 1$, $1 + k\nu > (1+k)^\nu$ for any positive number k. Therefore, an increase in $\tilde{\lambda}$ raises the value of V_K ($=$ the marginal value of capital q) if $\nu > 1$, and then accelererates investment. Otherwise, an increase in $\tilde{\lambda}$ decelerates investment. The uncertainty arising from the Poisson process also raises as well as depresses investment depending on the parameter value of φ. It is clear that when φ is close to one, uncertainty arising from the Poisson process raises investment.

In sum, we have arrived on the conclusion that, for three different cases of the demand shock, whether or not an increase in uncertainty raises investment depends on the convexity of the operating profit function with respect to the demand shock. For the operating profit function to be convex in the demand shock, the following inequality must be satisfied:

$$\varphi > 1 - \frac{\alpha}{\phi}.$$

Otherwise, the profit function is concave in the demand shock and an increase in uncertainty depresses investment. When $\varphi = 1 - \frac{\alpha}{\phi}$, an increase in uncertainty does not affect any effect on investment. When the product market is perfectly competitive, $\phi = 1$ and the condition for the convexity of the profit function reduces to $\varphi > 1 - \alpha$ [10].

3.2 The Case I with $\beta < 1$

We assume that $\beta < 1$, i.e. the product market is imperfectly competitive. We also assume that the demand shock is governed by the geometric Brownian motion (4). The Hamilton-Jacobi-Bellman equation for the marginal value of capital is given by Eq.(11):

$$\pi_K(K,X) - c_K(I^*,K) - \delta q + q_K(I^* - \delta K) + \mu(X)q_X + \frac{1}{2}\sigma(X)^2q_{XX} = \rho q.$$

When the marginal value of capital, q, is between the sale price of capital, b_2 and the purchase price of capital, b_1, it is optimal not to purchse nor to sell capital. It is optimal to purchase

[10] α is the share of labor and is arround $0.7 - 0.8$, which implies that $\varphi > 0.2 - 0.3$.

capital only when the marginal value of capital is greater than b_1 and to sell capital only when it is less than b_2. When $b_2 < q < b_1$, $I^* = 0$ so that the equation above is simplified into

$$\beta h X^{\nu} K^{\beta-1} - \delta q - q_K \delta K + \mu X q_X + \frac{1}{2}\sigma^2 X^2 q_{XX} = \rho q. \tag{28}$$

The general solution of eq.(28) can be expressed in the form

$$q(K, X) = AX^{\nu} K^{\beta-1} + B(X),$$

where the first term is the special solution and the second term is the homogenous solution. We suppose that the homogenous solution is in the form $B(X) = BX^{\theta}$. Then, θ must satisfy the following quadratic equation:

$$\frac{1}{2}\sigma^2\theta(\theta - 1) + \mu\theta - (\delta + \rho) = 0. \tag{29}$$

There exist two distinct roots of this equation, $\theta_1 > 1$, $\theta_2 < 0$. The value of $B(X)$ must remain finite when the value of X approaches zero, which means that the term of X^{θ_2} should vanishes. Therefore,

$$B(X) = BX^{\theta_1}.$$

The coefficient B is a constant that is yet to be determined. Substituting the special solution into eq.(28), we have

$$A = \frac{\beta h}{\rho + \beta\delta - \nu\mu - \frac{1}{2}\nu(\nu - 1)\sigma^2}.$$

The firm will undertake non-zero gross investment only if q reaches one of boundaries b_1 or b_2. The values of X at these boundaries, X_1, X_2, are given by the smooth-pasting and the high-contact conditions. The smooth-pasting conditions for the solution of eq.(28) are

$$q(K_1, X_1) = b_1,\ q(K_2, X_2) = b_2,$$

and the high-contact conditions are

$$\frac{\partial q(K_1, X_1)}{\partial X} = 0,\ \frac{\partial q(K_2, X_2)}{\partial X} = 0.$$

Using these boundary conditions gives rise to

$$X_1 = [\frac{b_1}{1 - \nu/\theta_1} \frac{\rho + \beta\delta - \nu\mu - \nu(\nu - 1)\sigma^2/2}{\beta h} K^{1-\beta}]^{1/\nu},$$

and

$$B = -\frac{\nu}{\theta_1}(\frac{b_1}{(1 - \nu/\theta_1)A})^{(\nu-\theta_1)/\nu} AK^{-(1-\beta)\theta_1/\nu}.$$

The trigger threshold to invest, X_1, is a function of b_1, θ_1, ν and the capital stock K.

Suppose that the profit function is concave in X, i.e. $\nu < 1$. Then,

$$\frac{b_1}{1-\nu/\theta_1}\frac{\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2}{\beta h} > 0$$

since $\nu/\theta_1 < 1$ and $\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2 > 0$ for $\theta_1 > \nu > \theta_2$. The trigger threshold X_1 is an increasing function of the direct cost of investment, i.e., the cost of purchasing capital goods. An increase in the purchase cost of capital goods raises the trigger level so that it would depress investment. If $1 \geq \beta+\nu$, an increase of the existing capital stock decreases investment. The inequality $1 \geq \beta+\nu$ holds when the operating profit function is linearly homogenous in X and K. This is an intuitively reasonable implication.What happens when uncertainty increases? An increase in σ increases the value of $\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2$. Since

$$\frac{d\theta_1}{d\sigma} = -\frac{2\sigma\theta_1(\theta_1-1)}{\sigma^2(\theta_1-1/2)+\mu} < 0,$$

an increase in σ decreases the value of θ_1 and so increases the value of $\frac{b_1}{1-\nu/\theta_1}$. Thus, an increase in uncertainty exerts a positive effect on the trigger threshold to invest.

The critical level of the operating profit corresponding to the trigger threshold to invest is given by

$$\pi(K, X_1) = hX_1^{\nu}K^{\beta} = \frac{b_1}{\beta}\frac{\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2}{1-\nu/\theta_1}K.$$

$\pi(K, X) > \pi(K, X_1)$ if and only if $X > X_1$ independently of the values of ν and β. The firm will undertake non-zero investment only when the operating profit $\pi(K, X)$ reaches the critical level $\pi(K, X_1)$. The critical trigger level of the profit is linearly homogenous in the existing capital stock. An increase in the capital stock induces in one-to-one a rise in the critical profit level. It should be reminded that $\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2$ is an increasing function of σ if $\nu < 1$ and that θ_1 is a decreasing function of σ. Therefore, an increase in uncertainty depresses investment and so there exists a negative relationship between uncertainty and investment as far as $\nu < 1$, i.e., the profit function is concave function in X.

Next we suppose that the operating profit function is convex in the demand shock, i.e., $\nu > 1$. In this case, $\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2$ is a decreasing function of σ. An increase in σ decreases the value of $\rho+\beta\delta-\nu\mu-\nu(\nu-1)\sigma^2/2$, so that a rise of uncertainty exerts a negative effect on the trigger threshold to invest. On the other hand, an increase in σ decreases the value of θ_1 and so it increases the value of $\frac{b_1}{1-\nu/\theta_1}$. Thus, an increase in uncertainty exerts two opposing effects on the trigger threshold to invest. A rise of uncertainty may decrease as well as raise the critical trigger level of the profit to invest, depending on the values of such parameters as ν and β. Therefore, when $\nu > 1$, there exists no simple monotonic relationship between uncertainty and investment.

The relationship between uncertainty and investment discussed above is based on the effect which unceratainty exerts on investment through changes in the trigger threshold to invest. We have not analyzed the traditional famous effect that changes in the marginal value of capital cause the amount of investment when positive gross investment is being undertaken. We will explore this issue. When the marginal value of capital is greater than the trigger threshold X_1, optimal investment is given

$$I^* = \left(\frac{q-b_1}{\eta_1\gamma_1}\right)^{1/(\gamma_1-1)}.$$

The Hamilton-Jacobi equation is

$$\beta h X^{\nu} K^{\beta-1} - \delta q + q_K\{(\frac{q-b_1}{\eta_1\gamma_1})^{1/(\gamma_1-1)} - \delta K\} + \mu X q_X + \frac{1}{2}\sigma^2 X^2 q_{XX} = \rho q. \tag{30}$$

This partial differential equation is difficult to solve analytically.

3.3 The Case II

We consider the case II, in which $\gamma_2 \neq 0$. The cost function of investment is given by

$$c(I,K) = \begin{cases} a_1K + b_1I + \eta_1 I^{\gamma_1}K^{\gamma_2}, & \text{when } I > 0 \\ 0, & \text{when } I = 0, \\ a_2K + b_2I + \eta_2|I|^{\gamma_1}K^{\gamma_2}, & \text{when } I < 0 \end{cases}.$$

Optimal investment is characterized by

$$\begin{array}{ll} (I^*)^{\gamma_1-1} = \frac{q-b_1}{\eta_1\gamma_1K^{\gamma_2}} > 0, & \text{when } q > b_1 = c_I(0,K)^+, \\ I \quad = \quad 0, & \text{when } b_1 \geq q \geq b_2 = c_I(0,K)^-, \\ (-I^*)^{\gamma_1-1} = -\frac{q-b_2}{\eta_2\gamma_1K^{\gamma_2}} > 0, & \text{when } b_2 > q \end{array}$$

When the marginal value of capital, q, is between the sale price of capital (b_2) and the purchase price of capital (b_1), it is optimal not to purchse nor to sell capital. It is optimal to buy capital only if q reaches the boundary b_1 and to sell capital only if q reaches the boundary b_2. We assume that the demand shock is governed by the geometric Brownian motion (4). The Hamilton-Jacobi-Bellman equation for the marginal value of capital, q, is given by Eq.(11):

$$\pi_K(K,X) - c_K(I^*,K) - \delta q + q_K(I^* - \delta K) + \mu(X)q_X + \frac{1}{2}\sigma(X)^2 q_{XX} = \rho q.$$

When $I^* = 0$, the equation above is simplified into

$$\beta h X^{\nu} K^{\beta-1} - \delta q - \delta q_K K + \mu X q_X + \frac{1}{2}\sigma^2 X^2 q_{XX} = \rho q. \tag{31}$$

The general solution of eq.(31) can be expressed in the form

$$q(K,X) = AX^{\nu}K^{\beta-1} + B(X),$$

where the first term is the special solution and the second term is the homogeneous solution. We suppose that the homogeneous solution is in the form $B(X) = BX^{\theta}$. We suppose that the homogeneous solution is in the form $B(X) = BX^{\theta}$. Then, θ must satisfy the following quadratic equation:

$$\frac{1}{2}\sigma^2\theta(\theta-1) + \mu\theta - (\delta+\rho) = 0. \tag{32}$$

There exist two distinct roots of this equation, $\theta_1 > 1$, $\theta_2 < 0$. The value of $B(X)$ must remain finite when the value of X approaches zero, which means that the term of X^{θ_2} should vanishes. Therefore,

$$B(X) = BX^{\theta_1}.$$

The coefficient B is a constant that is yet to be determined. Substituting the special solution into eq.(31), we have

$$A = \frac{\beta h}{\rho + \beta\delta - \nu\mu - \frac{1}{2}\nu(\nu-1)\sigma^2}.$$

The firm will undertake non-zero gross investment only if q reaches one of boundaries b_1 or b_2. The values of X at these boundaries, X_1, X_2, are given by the smooth-pasting and the high-contact conditions. The same analysis as undertaken in the previous section can be used here. Therefore, the basically same result concerning with the relationship between uncertainty and the trigger threshold to invest will apply. An increase in uncertainty depresses investment and so there exists a negative relationship between uncertainty and investment as far as $\nu < 1$, i.e., the profit function is concave function in X. On the other hands, when $\nu > 1$, the negative relationship between uncertainty and investment does not necessarily hold.

4 Concluding Remarks

We have shown that an increase in uncertainty decelerates as well as accelerates investment depending on the value of the model paratemets. In particular, as far as the operating profit function is concave in the demand shock, an increase in uncertainty depresses investment, but even if the operating profit function is convex in the demand shock, an increase in uncertainty may not necessarily raise investment and would depress investment, depending on the range of model parameters. This result implies that the convexity of the operating profit function need not necesarily to be dismissed in order to be consistent with the emprirical validity.

Reference

(1). Andrew B. Abel(1983), Optimal Investment under Uncertainty, *American Economic Review*, 73(1), 228-233.

(2). Andrew B. Abel(1985), A Stochastic Model of Investment, Marginal q and the Market Value of the Firm, *International Economic Review*, 26(2), 305-322.

(3). Andrew B. Abel and Janice C. Eberly(1994), A Unified Model of Investment Under Uncertainty, *American Economic Review*, 84(1).

(4). Andrew B. Abel and Janice C. Eberly(1996), Optimal Investment with Costly Reversibility, *Review of Economic Studies*, 63, 581-593.

(5). Andrew B. Abel and Janice C. Eberly(1999), The Effects of Reversibility and Uncertainty on Capital Accumulation, *Journal of Monetary Economics*, 44, 339-377.

(6). Ben S. Bernanke(1983), Irreversibility, Uncertainty, and Cyclical Investment, *Quarterly Journal of Economics*, 98(1), 85-106.

(7). Guiseppe Bertola and Ricardo J. Caballero(1994), Irreversibility and Aggregate Investment, *Review of Economic Studies*, 61, 223-246.

(8). Michael J. Brennan and Eduardo S. Schwartz(1985), Evaluating Natural Resources Investment, *Journal of Business*, 58(2), 135-157.

(9). Ricardo J. Caballero(1991), On the Sign of the Investment-Uncertainty Relationship, *American Economic Review*, 81(1),279-288.

(10). Ricardo J. Caballero and Robert S. Pindyck(1996), Uncertainty, Investment, and Industry Evolution, *International Economic Review*, 37(3), 641-662.

(11). Avinash K. Dixit and Robert S. Pindyck(1994), *Investment Under Uncertainty*, Princeton University Press.

(12). Sebatian Gryglewicz, Kuno J. M. Huisman, and Peter M. Kort(2008), Finite Project Life and Uncertainty Effects on Investment, *Journal of Economic Dynamics and Control*, 32, 2191-2213.

(13). Richard Hartman(1972), The Effects of Price and Cost Uncertainty on Investment, *Journal of Economic Theory*, 5, 258-266.

(14). Ioannis Karatzas and Steven E. Shreve(1991), *Brownian Motion and Stochastic Calculus*, second edition, Springer-Verlag.

(15). John V. Leahy and Toni M. Whited(1996), The Effects of Uncertainty on Investment: Some Stylized facts, *Journal of Money, Credit, and Banking*, 28(1), 64-83.

(16). Robert McDonald and Daniel Siegel(1985), Investment and the Valuation of Firms When There is an Option to Shut Down, *International Economic Review*, 26(2), 331-349.

(17). Robert McDonald and Daniel Siegel(1986), The Value of Waiting to Investment, *Quarterly Journal of Economics*, 101(4), 707-728.

(18). Marek Musiela and Marek Rutkowski(2007), *Martingale Methods in Financial Modelling*, Springer.

(19). Bernt Øksendal(2007), *Stoshatic Differential Equations*, sixth edition, Springer-Verlag.

(20). Robert S. Pindyck(1988), Irreversible Investment, Capacity Choice, and the Value of the Firm, *American Economic Review*, 78(5), 969-985.

(21). Robert S. Pindyck(1991), Irreversibility, Uncertainty, and Investment, *Journal of Economic Literature*, 29(3), 1110-1148.

(22). Robert S. Pindyck(1993), A Note on Competitive Investment under Uncertainty, *American Economic Review*, 83(1), 273-277.

(23). Sudipto Sarkar(2000), On the Investment-Uncertainty Relationship in a Real Options Model, *Journal of Economic Dynamics and Control*, 24、219-225.

(24). Steven E. Shreve(2004), *Stochastic Calulus for Finance II: Continuous-Time Models*, Springer.

(25). Kit Pong Wong(2007), The Effect of Uncertainty on Investment Timing in a Real Options Model, *Journal of Economic Dynamics and Control*, 31, 2152-2167.